Name: ______________________________ Section: 201 204

CIS260-201/204–Spring 2008

Recitation 12 Supplementary Exercises

Friday, April 18

1. David tosses a fair coin eight times. Given that his first and last outcomes are the same, what is the probability he tossed five heads and three tails?

2. A grab bag contains one chip with the number 1, two chips each with the number 2, three chips each with the number 3, ..., and n chips each with the number n, where $n \in \mathbb{Z}^+$. All chips are of the same size, those numbered 1 to m are red, and those numvered $m + 1$ to n are blue, where $m \in \mathbb{Z}^+$ and $m \leq n$. If Dale draws one chip, what is the probability it is the chip with 1 on it, given that the chip is red?

3. Let X be a random variable defined on a sample space $(S, \Pr)$. Is it possible for X to be independent of itself?

4. Let A and B be events. Prove that $\Pr(A \mid B) \geq \frac{\Pr(A)+\Pr(B)-1}{\Pr(B)}$.

5. An unfair coin shows HEADS with probability p and TAILS with probability $1-p$. Suppose this coin is tossed twice. Let A be the event that the coin comes up first HEADS and then TAILS and let B be the event that the coin comes up first TAILS and then HEADS.

   (a) Calculate $\Pr(A)$.

   (b) Calculate $\Pr(B)$.

   (c) Calculate $\Pr(A \mid A \cup B)$.

   (d) Calculate $\Pr(B \mid A \cup B)$.

   (e) Explain how to use an unfair coin to make a fair decision (choose between two alternatives with equal probability).

6. When a coin is tossed three times, for the outcome HHT we say that two *runs* have occurred, namely, HH and T. Likewise, for the outcome THT we find three runs: T, H, and T. Now suppose a biased coin, with $\Pr(H) = 3/4$, is tossed three times and the random variable X counts the number of runs that result.

   (a) Determine the probability distribution for X. [**Hint**: What are the possible outcomes of X?]

   (b) Determine $\mathrm{E}[X]$.

7. Let X be a random variable with probability distribution

$$\Pr(X = x) = \begin{cases} c(x^2 + 4), & x = 0, 1, 2, 3, 4 \\ 0, & \text{otherwise,} \end{cases}$$

where c is a constant. Determine

   (a) the value of c

   (b) $\Pr(X > 1)$

   (c) $\Pr(X = 3 \mid X \geq 2)$ [**Hint**: Use part 7(b).]

   (d) $\mathrm{E}[X]$

8. On the way from 30th Street Station to 36th Street Station, a SEPTA Route-34 trolley encounters eight signal posts. One day the signal system goes awry and each post displays a Red light 25% of the time, independent of other signal posts. Chin loves looking toward the front of the trolley and he happens to be on this trolley that day. If the random variable X counts the number of Red lights Chin encounters on his way from 30th Street Station to 36th Street Station, determine

   (a) $\Pr(X = 0)$

   (b) $\Pr(X = 3)$

   (c) $\Pr(X \geq 6)$

   (d) $\Pr(X \geq 6 \mid X \geq 4)$

   (e) $\mathrm{E}[X]$

9. Suppose X is a zero-one random variable, i.e., an indicator random variable. Prove that $E[X] = E[X^2]$.

10. Let X and Y be real-valued random variables defined on a sample space $(S, \Pr)$. Suppose $X(s) \leq Y(s)$ for all $s \in S$. Prove that $E[X] \leq E[Y]$.

11. In this problem we explore the *Markov's inequality*. Let $(S, \Pr)$ be a sample space and let $X : S \to \mathbb{N}$ be a nonnegative-integer-valued random variable. Let a be a positive integer. Prove that

$$\Pr(X \geq a) \leq \frac{E[X]}{a}.$$

[**Hint**: $\Pr(X \geq a) = \Pr(X = a) + \Pr(X = a + 1) + \Pr(X = a + 2) + \cdots$.]

12. In class on R 04/17 we stated a theorem about variance: Let X and Y be random variables. If X and Y are independent, then

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

Prove this theorem. [**Hint**: If X and Y are independent, then $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$.]