11 Name: ______________________ Section: 201 204

CIS260-201/204–Spring 2008

Recitation 11 Supplementary Exercises

Friday, April 11

First Note: To be a pro in probability theory, you need to do a lot of problems. The more you expose yourself to problems, the easier you are able to interpret them. Some of the following exercises are from the textbook, but I recommend that you do the remaining exercies in the book as well, especially the computational ones. They will not take too much time, in contrast with those problems on graph theory. I will try to give you problems from some other sources as well.

1. A fair coin is flipped ten times.

   (a) What is the probability that there are an equal number of HEADS and TAILS?

   (b) What is the probability that the first three flips are HEADS?

   (c) What is the probability that there are an equal number of HEADS and TAILS and the first three flips are HEADS?

   (d) What is the probability that there are an equal number of HEADS and TAILS or the first three flips are HEADS (or both)?

   (e) Are the two events in parts 1(a) and 1(b) independent? In any case, what is $\Pr(A \mid B)$ and $\Pr(B \mid A)$, where A is the event in part 1(a) and B is the event in part 1(b)?

2. Suppose A and B are events in a sample space. Prove or disprove:

   (a) If $A \subseteq B$, then $\Pr(A) \leq \Pr(B)$.

   (b) If $A \subsetneq B$, then $\Pr(A) < \Pr(B)$.

3. Let A and B be events in a sample space $(S, \Pr)$ and suppose $0 < \Pr(B) < 1$, prove that $\Pr(A \mid B) \Pr(B) + \Pr(A \mid \overline{B}) \Pr(\overline{B}) = \Pr(A)$.

4. Give an example of an event B such that $\Pr(A \mid B) \Pr(B) = \Pr(A)$.

5. Let A and B be events in a sample space. Prove or disprove the following statements.

   (a) If A and B are independent, then A and $\overline{B}$ are independent.

   (b) If A and B are independent, then $\overline{A}$ and $\overline{B}$ are independent.

6. Let A and B be events in a sample space. Prove or disprove:

   (a) If $\Pr(A) = 0$, then A and B are independent.

   (b) If $\Pr(A) = 1$, then A and B are independent.

7. Let A, B be events taken from a sample space Ω (with $\Pr(A) > 0$ and $\Pr(B) > 0$). If $\Pr(B \mid A) < \Pr(B)$, prove that $\Pr(A \mid B) < \Pr(A)$.

8. *More card problem*: Recall that in a standard deck of 52 cards, there are 12 picture cards—four each of jacks, queens, and kings. Donald draws one card from the deck. Find the probability that his card is a king if we know that the card drawn is an ace or a picture card.

9. Three hockey players are doing penalty shootouts. The probabilities that the individual players 1, 2, and 3 will score are 0.75, 0.85, and 0.9, respectively. Find the probability that at least two of the players score.

10. Let A, B be events taken from a sample space Ω. If $\Pr(A \cap B) = 0.1$ and $\Pr(\overline{A} \cap \overline{B}) = 0.3$, what is $\Pr(A \,\Delta\, B \mid A \cup B)$?

11. Consider the following problem:

    While traveling through Pennsylvania, Gus the Groundhog decides to buy a lottery ticket for which he selects seven integers from 1 to 80, inclusive. The state lottery commission then selects 11 of these 80 integers. If Gus's selection matches seven of these 11 integers, he is a winner. What is the probability that Gus is winner (and so he stops showing up on TV)?

    Consider the following two solutions:

    (a) Gus can select 7 numbers from 80 integers in $\binom{80}{7}$ ways. The commission has a set of 11 winning numbers. If Gus's numbers are among these 11 numbers, he is a winner. There are $\binom{11}{7}$ such possible combinations. Therefore, the probability that he is a winner is $\frac{\binom{11}{7}}{\binom{80}{7}}$.

    (b) The state lottery commission can select 11 numbers from 80 integers in $\binom{80}{11}$ ways. Gus has a set of 7 numbers. If Gus is to be a winner, then these 7 numbers must be a subset of those 11 numbers. Hence, the commission can select 4 other numbers from the set of numbers that Gus did not select, which can be done in $\binom{80-7}{4} = \binom{73}{4}$ ways. Therefore, the probability that Gus is a winner is $\frac{\binom{73}{4}}{\binom{80}{11}}$.

    Which solutions are correct? For the incorrect solutions, argue why they are so.

12. Let $(S, \Pr)$ be the sample space representing n tosses of a fair coin. Let $X : S \to \mathbb{Z}$ be the random variable that gives the number of HEADS minus the number of TAILS. Determine all $x \in \mathbb{Z}$ such that there exists an outcome $s \in S$ such that $X(s) = x$. When is $x = 0$ among the possible x's?

13. (a) In class on R 04/10 we derived the expected number of heads from 8 tosses of a fair coin to be

$$\sum_{i=0}^{8} \frac{i\binom{8}{i}}{2^8} = \frac{0\binom{8}{0}}{2^8} + \frac{1\binom{8}{1}}{2^8} + \cdots + \frac{8\binom{8}{8}}{2^8}.$$

Simplify this expression and conclude that it equals 4. [**Hint**: You don't need to expand the binomial coefficients to simplify this. How do you derive the formula for the sum of the first n positive integers?]

(b) [Extra Credit!] We also showed that if we toss a biased coin with probability p of coming up head 8 times, then the expected number of heads is

$$\sum_{i=0}^{8} i\binom{8}{i} p^i (1-p)^{8-i}.$$

Simplify this expression and conclude that it equals $8p$. [**Hint**: Since this will take some space, please attach extra sheets if you would like to do this problem. This time you will need to expand the binomial coefficients.]