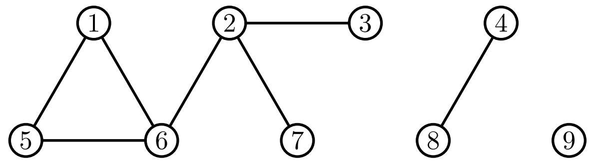
 Name: ______________________ Section: 201 204

# CIS260-201/204–Spring 2008

## Recitation 8 Supplementary Exercises

Friday, March 21

1. Consider the following graph $G$.



(a) Describe $V$ and $E$ of $G$.

$V =$

$E =$

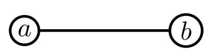
(b) What is the degree of each vertex in $G$?

| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Degree | | | | | | | | | |

(c) How many edges are there in $G$? What is the sum of the degrees? Is there any relation between these two numbers? If so, is the relation true in general?

(d) How many connected components are there in $G$? List each connected component.

2. Consider the following graph.



(a) How many paths are there from $a$ to $b$? List them.

(b) How many walks are there from $a$ to $b$? List them (?).

3. Let $G = (V, E)$ be a graph. Describe all the spanning subgraphs of $G$ that are also induced subgraphs of $G$.

4. In this exercise, we explore a special kind of graphs called *complete graph*.
**Definition**: A graph $G(V, E)$ is complete if for any vertices $u, v \in V$, if $u$ and $v$ are distinct, then $u$ and $v$ are adjacent.
**Notation**: We denote a complete graph of $n$ vertices as $K_n$.

   (a) Write the definition of complete graph (the part after *if* above) using quantifiers. There should be no words in your statement. Only symbols are allowed. [**Hint**: What does it mean to be adjacent?]

   (b) Describe, using only quantifiers and symbols, when a graph is *not* complete.

   (c) What is the degree of each vertex in $K_n$? Explain.

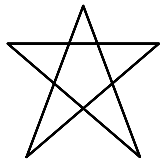
   (d) How many edges are there in $K_n$? Explain.

   (e) How many colors do we need to color $K_n$ so that the coloring is valid (as defined before)? Explain.

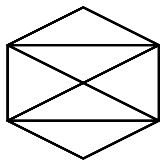
   (f) [Extra credit!] Can you guess $n$ such that $K_n$ has an Euler tour?

5. (a) For each of the figures below, try to draw it *without* lifting your pencil. Which ones can you do so? For those that you can, which ones can you start and end at the same point? (Of course, if you can do it, draw the figure to the right of the original figure.)

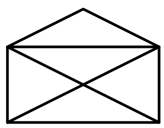
i.



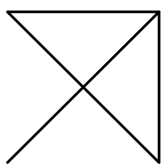
ii.



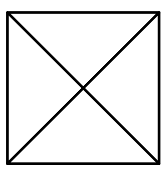
iii.



iv.



v.



(b) [Extra credit!] What can you say in general about the figures that you can draw without lifting your pencil? What can you say in general about the figures that you can draw without lifting your pencil *and* by starting and ending at the same point? Is there anything related to what we have done so far?

6. On R 03/20 we had a quite serious discussion in lecture about using induction to prove properties of trees. Now that we have enough tools to do so, we will go ahead and use induction whenever we can! (That's cool. :D) [**Hint**: If you don't see the answer right away, try some examples and generalize the results. This technique is useful for most of the proofs. One note: Intuition is useful, but don't let it fool you in the end; that's why we *prove*.]

   (a) Let $T$ be a tree of at least 2 nodes (by the way, nodes are just the same as vertices; we use them interchangeably), what is the minimum number of leaves $T$ must have? Prove this by induction.

   (b) Let $T$ be a tree of $n$ nodes, where $n \geq 3$. What is the maximum number of leaves $T$ may possibly have? Prove this by induction.

7. Proving an inequality by induction is a little (or much?) more difficult than proving an equality. So far we have not done much on proving inequalities by induction. Now we will do so. The key point is to recognize that some terms in the expression are less than (or greater than) the desired result (with some reasoning) and proceed to that goal.

   Recall the Fibonacci numbers $F_n$, where $n \in \mathbb{N}$, defined as follows:

$$F_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases} .$$

   (a) Prove by induction that $F_n \leq F_{n+1}$ for all $n \geq 0$. (Be careful about which formula holds for which $n$.)

   (b) Prove by strong induction that $F_n \leq 1.7^n$ for all $n \geq 0$.

(c) Prove that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}},$$

where $n$ is any positive integer. Briefly explain the reason you reach each step in your proof. [**Hint**: Try it somewhere else before writing your final proof below; you might need to scratch a little bit (or a lot).]