CIS260-201/204–Spring 2008

Inverse of a Function

Friday, February 22

We have studied functions in class. This document recapitulates some definitions needed for later discussion in the document about the inverse of a function.

1 Functions

Definition: Let A and B be sets. f is a *function from A to B*, denoted $f : A \to B$ if

1. f is a relation from A to B, i.e., $f \subseteq A \times B$,
2. for every element $a \in A$ and any elements $b_1, b_2 \in B$, if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$, and
3. for every element $a \in A$, there is an element $b \in B$ such that $(a, b) \in f$.

Property 1 states that in order for f to be a function from A to B, f must at first be a relation. Property 2 states that any element $a \in A$ relates to at most one element in B. These two properties comprise the definition of *function*. Property 3 states that any element $a \in A$ relates to at least one element in B. Therefore, the last two properties state that every element in A relates to exactly one element in B. All the three properties comprise the definition of $f : A \to B$.

Notation: When f is a function and $(a, b) \in f$, we write $f(a) = b$.

Definition: $f : A \to B$ is *one-to-one* if for any elements $a_1, a_2 \in A$ and $b \in B$, if $(a_1, b) \in f$ and $(a_2, b) \in f$, then $a_1 = a_2$.
Since f is a function, we can also write, "if $f(a_1) = f(a_2)$, then $a_1 = a_2$." This definition states that any element $b \in B$ has at most one element in A that relates, or maps, to it. Notice that this definition is quite similar to property 2 in the definition of functions.

Definition: $f : A \to B$ is *onto* if for any element $b \in B$, there is an element $a \in A$ such that $(a, b) \in f$.
Since f is a function, we can also write, "$f(a) = b$." This definition states that any element $b \in B$ has at least one element in A that maps to it. Again, notice that this definition is quite similar to property 3 in the definition of functions.

Definition: $f : A \to B$ is a *bijection* if f is one-to-one and onto.
This definition states that any element in A relates to exactly one element in A. Notice that this definition is quite similar to the combination of properties 2 and 3 in the definition of functions.

2 Inverse of a Function

Definition: Let R be a relation. The *inverse* if R, denoted R^{-1}, is defined by

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

That is, the inverse of a relation R is obtained by flipping the elements in each ordered pair of R.

If f is a function from A to B. f might not be a function.
Example: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. Consider $f = \{(1, 2), (2, 3), (3, 2)\}$. Observe that f is a function from A to B. Now,

$$f^{-1} = \{(2, 1), (2, 3), (3, 2)\}.$$

It is easy to see that f^{-1}—the inverse of f—is not a function because an element in the first set maps to more than one element in the second set.

To avoid the first coordinate in f^{-1} mapping to more than one element in the second set, no two elements in A should map to the same element in the second set B. That is, f must be one-to-one in the first place.
Example: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. Consider $f = \{(1, 2), (2, 3), (3, 4)\}$. Observe that f is a function from A to B. It follows that

$$f^{-1} = \{(2, 1), (3, 2), (4, 3)\}.$$

Now f^{-1} is a function, but *not* a function from B to A because $1 \in B$ maps to nothing in A.

To ensure that the first coordinate in f^{-1} map to at least one element in the second set, there should be at least one element in A that map to each element in B. That is, f must be onto in the first place.

Now we see that if f is a function from A to B, in order for f^{-1} to be a function, f must be one-to-one. In order for f^{-1} to be a function from B to A, f must be one-to-one and onto. This leads to the following theorem, which we state here without a proof.

Theorem: Let $f : A \to B$. f^{-1} is a function from B to A if and only if f is one-to-one and onto.

3 Example

Prove that if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection.
Proof: We need to show that f^{-1} is a function from B to A that is a bijection. That is, we need to show that

1. f^{-1} is a function from B to A,
2. f^{-1} is one-to-one, and
3. f^{-1} is onto.

To prove 1, observe that by the theorem stated above, since f is a bijection, f^{-1} is a function from B to A, and we are done for this.

To prove 2, we need to show that for any elements $b_1, b_2 \in B$ and $a \in A$, if $(b_1, a) \in f^{-1}$ and $(b_2, a) \in f^{-1}$, then $b_1 = b_2$. That is, by the definition of the inverse of a function, for every

$a \in A$ and any elements $b_1, b_2 \in B$, if $(a, b_1) \in f$ and $(a, b_2) \in f$. But this is exactly property 2 in the definition of a function. Since f is a function, f has this property, and f^{-1} is one-to-one as desired.

To prove 3, we need to show that for any element $a \in A$, there is an element $b \in B$ such that $(b, a) \in f^{-1}$. That is, by the definition of the inverse of a function, for any element $a \in A$, there is an element $b \in B$ such that $(a, b) \in f$. Since f is a function, it has all the three properties in the definition of a function. But the previous statement is exactly property 3. Therefore, f^{-1} is onto, as required.

Finally, since we have proven all the three cases, we conclude that if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection. □