CIS121-204–Fall 2007

Lab 3 Solution

Tuesday, September 25

1 Mathematical Induction

1.1 Exercises

1. Show that $\sum_{i=1}^{n} 2i - 1 = 1 + 3 + \cdots + (2n - 1) = n^2$ for all $n \geq 1$.
   Base case ($n = 1$): $\sum_{i=1}^{n} 2i - 1 = \sum_{i=1}^{1} 2i - 1 = 2 - 1 = 1$, which is true.
   Inductive step: Assuming that $\sum_{i=1}^{k} 2i - 1 = k^2$, show that $\sum_{i=1}^{k+1} 2i - 1 = (k+1)^2$. Since $\sum_{i=1}^{k+1} 2i - 1 = \sum_{i=1}^{k} 2i - 1 + (2(k + 1) - 1)$, by the inductive hypothesis, we have

$$\begin{aligned} \sum_{i=1}^{k+1} 2i - 1 &= k^2 + 2(k + 1) - 1 \\ &= k^2 + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

   as required. Hence, we have proven that $\sum_{i=1}^{n} 2i - 1 = n^2$ for all $n \geq 1$. □

2. Show that $\sum_{i=1}^{n} (i)(i!) = (n + 1)! - 1$ for all $n \geq 1$.
   Base case ($n = 1$): $\sum_{i=1}^{n} (i)(i!) = \sum_{i=1}^{1} (i)(i!) = 1(1!) = 1$ and $(1+1)! - 1 = 2! - 1 = 1$.
   Inductive step: Assuming that $\sum_{i=1}^{k} (i)(i!) = (k + 1)! - 1$, show that $\sum_{i=1}^{k+1} (i)(i!) = (k + 2)! - 1$. Since $\sum_{i=1}^{k+1} (i)(i!) = \sum_{i=1}^{k} (i)(i!) + (k + 1)(k + 1)!$, by the inductive hypothesis, we have

$$\begin{aligned} \sum_{i=1}^{k+1} (i)(i!) &= (k + 1)! - 1 + (k + 1)(k + 1)! \\ &= (k + 1)!\,(1 + (k + 1)) - 1 \\ &= (k + 2)(k + 1)! - 1 \\ &= (k + 2)! - 1 \end{aligned}$$

   as required. Hence, we have proven that $\sum_{i=1}^{n} (i)(i!) = (n + 1)! - 1$ for all $n \geq 1$. □

3. Show that $n! > 2^{n-1}$ for all $n \geq 3$.
   Base case ($n = 3$): $3! = 6$ and $2^{3-1} = 2^2 = 4$. Hence, $6 > 4$, and the base case is proven.
   Inductive step: Assuming that $k! > 2^{k-1}$, show that $(k + 1)! > 2^k$. Since $(k + 1)! = (k + 1)k!$, by the inductive hypothesis, we have

$$\begin{aligned} (k + 1)! &> (k + 1)2^{k-1} \\ &> 2 \cdot 2^{k-1} \text{ (because } k + 1 > 2 \text{ for all } k \geq 3) \\ &= 2^k \end{aligned}$$

as required. Hence, we have proven that $n! > 2^{n-1}$ for all $n \geq 3$. $\square$

4. Show that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\ldots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}}$ for all $n \geq 1$.
   Base case ($n = 1$): $\left(1-\frac{1}{2}\right) = \frac{1}{2}$ and $\frac{1}{4}+\frac{1}{2^2} = \frac{1}{2}$. Hence, $\frac{1}{2} \geq \frac{1}{2}$, and the base case is proven.
   Inductive step: Assuming that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\ldots\left(1-\frac{1}{2^k}\right) \geq \frac{1}{4}+\frac{1}{2^{k+1}}$, show that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\ldots\left(1-\frac{1}{2^{k+1}}\right) \geq \frac{1}{4}+\frac{1}{2^{k+2}}$. Since
   $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\ldots\left(1-\frac{1}{2^{k+1}}\right) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\ldots\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k+1}}\right)$,
   by the inductive hypothesis, we have

$$\begin{aligned}
\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\ldots\left(1-\frac{1}{2^{k+1}}\right) &\geq \left(\frac{1}{4}+\frac{1}{2^{k+1}}\right)\left(1-\frac{1}{2^{k+1}}\right) &(1)\\
&= \frac{1}{4}+\frac{1}{2^{k+1}}-\frac{1}{4\cdot 2^{k+1}}-\frac{1}{2^{2k+2}} &(2)\\
&= \frac{1}{4}+\frac{3}{4\cdot 2^{k+1}}-\frac{1}{2^{2k+2}} &(3)\\
&= \frac{1}{4}+\frac{3}{2\cdot 2^{k+2}}-\frac{1}{2^{2k+2}} &(4)\\
&\geq \frac{1}{4}+\frac{1}{2^{k+2}}\left(\frac{3}{2}-\frac{1}{2^k}\right) &(5)\\
&\geq \frac{1}{4}+\frac{1}{2^{k+2}}\left(\frac{3}{2}-\frac{2^{k-1}}{2^k}\right) &(6)\\
&= \frac{1}{4}+\frac{1}{2^{k+2}}\left(\frac{3}{2}-\frac{1}{2}\right) &(7)\\
&= \frac{1}{4}+\frac{1}{2^{k+2}} &(8)
\end{aligned}$$

   as required. In Equation (6), we used the fact that $k \geq 1$, so $2^{k-1} \geq 1$. Hence, we have proven that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\ldots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}}$ for all $n \geq 1$. $\square$

5. Show that $n^2 + n$ is divisible by 2 for all $n \geq 1$.
   Base case ($n = 1$): $1^2 + 1 = 2$, which is divisible by 2.
   Inductive step: Assuming that $k^2 + k$ is divisible by 2, show that $(k+1)^2 + (k+1)$ is divisible by 2. Now,

$$\begin{aligned}
(k+1)^2 + (k+1) &= k^2 + 2k + 1 + k + 1\\
&= k^2 + k + 2k + 2
\end{aligned}$$

   By the inductive hypothesis, $k^2 + k$ is divisible by 2, and $2k + 2$ is obviously divisible by 2. Hence, the sum of the two terms, $k^2 + k + 2k + 2$, is divisible by 2. Hence, we have proven that $n^2 + n$ is divisible by 2 for all $n \geq 1$. $\square$
   Note that $n^2 + n = n(n+1)$, the product of two consecutive integers. Therefore, one of them must be even, and the product must be divisible by 2. In this case, we need not prove by induction.

6. Show that $2^{2n} - 1$ is divisible by 3 for all $n \geq 1$.
   Base case ($n = 1$): $2^2 - 1 = 4 - 1 = 3$, which is divisible by 3.
   Inductive step: Assuming that $2^{2k} - 1$ is divisible by 3, show that $2^{2(k+1)} - 1$ is divisible by 3. Now,

$$\begin{aligned} 2^{2(k+1)} - 1 &= 4 \cdot 2^{2k} - 1 \\ &= 4 \cdot 2^{2k} - 4 + 3 \\ &= 4\left(2^{2k} - 1\right) + 3 \end{aligned}$$

   By the inductive hypothesis, $2^{2k} - 1$ is divisible by 3, so $4\left(2^{2k} - 1\right)$ is divisible by 3. Obviously, 3 is divisible by 3, and therefore the sum of the two terms, $4\left(2^{2k} - 1\right) + 3$, is divisible by 3. Hence, we have proven that $2^{2n} - 1$ is divisible by 3 for all $n \geq 1$. □

1.2 What Went Wrong?

Show that all horses in the world have the same color.

Base case: If there is only one horse in the world, it is obvious that one horse has the same color.

Inductive step: Assume that k horses have the same color. We will show that $k + 1$ horses have the same color. Given $k + 1$ horses, we remove one of them. Hence, there are k remaining horses, and by the inductive hypothesis, all of them have the same color. Now we remove another horse and replace the first one; there are still k horses, so they all have the same color. Repeat this process $k + 1$ times until every horse is picked. It follows that since every horse has the same color as every other horse, all the $k + 1$ horses have the same color.

Hence, we have just proven that all horses in the world have the same color. ■

So what went wrong here? Consider when there are two horses; we are in the inductive case. We remove one of the two horses. Hence, there is 1 remaining horse, and it has a color. Now we put that horse back and remove the other horse. There is still 1 horse, but we cannot conclude this horse has the same color as the other one. Hence, the inductive step fails at $n = 2$, i.e., it is true for all $n \geq 3$. In order to prove the statement, we need to have $n = 2$ as the base case. But now we can go to a barn and pick two horses with different colors to disprove the base case. Therefore, our argument fails completely. □

2 Big-O Notation

2.1 Exercises

1. Show that $n^2 + 4n + 3$ is $O(n^3)$.
   Even though we can easily prove this using the Big-O theorems stated in class, here we are going to use the definition of Big-O.

   We want to find a positive real constant c and a natural number N such that for all $n \geq N$,

$n^2 + 4n + 3 \leq cn^3$. Now observe that

$$\begin{aligned} n^2 + 4n + 3 &\leq n^2 + 4n + 3n \text{ for all } n \geq 1 \\ &= n^2 + 7n \text{ for all } n \geq 1 \\ &\leq n^2 + 7n^2 \text{ for all } n \geq 1 \\ &= 8n^2 \text{ for all } n \geq 1 \\ &\leq n \cdot n^2 = n^3 \text{ for all } n \geq 8 \end{aligned}$$

Hence, by letting $c = 1$ and $N = 8$, we have proven that $n^2 + 4n + 3$ is $O(n^3)$. □
Note that there are many other solutions; this is just one of them.

2. Show that n is not $O(\log n)$.
   Prove by contradiction. Assume that n is $O(\log n)$. Then there exist c and N such that $n \leq c \log n$ for all $n \geq N$. Now, consider two cases:

   (a) $c \leq 4$: Then choose $n = \max(16, N) + 1$. Hence, $n \leq c \log n \leq 4 \log n$, which means $n \leq 4 \log n$, which is obviously false for all $n > 16$.

   (b) $c > 4$: Then choose $n = \max(c^2, 16, N) + 1$. Since $c > 2 \log c$ for all $c > 4$, we have

   $$\begin{aligned} c^2 &> 2c \log c \\ &= c \log c^2 \end{aligned}$$

   for all $c > 4$, i.e., $c^2 > 16$. But we assume that for any $n \geq N$, we must have $n \leq c \log n$. Hence, for $n = \max(c^2, 16, N) + 1$ we derive a contradiction.

   In any case, for any given $c > 0$ and N, we can find n such that $n > c \log n$. Therefore, n is not $O(\log n)$. □

Prove or disprove the following:

3. If $0 < f(n) < g(n)$ for all $n > 1000$, then $f(n) + g(n)$ is $O(g(n))$.
   Since $f(n) < g(n)$ for all $n > 1000$, then $f(n) + g(n) < g(n) + g(n) = 2g(n)$ for all $n > 1000$. By the definition of Big-O, it follows immediately that $f(n) + g(n)$ is $O(g(n))$ (where $c = 2$ and $N = 1001$). □

4. If $0 < f(n) < g(n)$ for all $n > 100$, then $n^2 f(n) + g(n)$ is $O(n^2 g(n))$.
   Since $f(n) < g(n)$ for all $n > 100$,

   $$\begin{aligned} n^2 f(n) + g(n) &< n^2 g(n) + g(n) \\ &< n^2 g(n) + n^2 g(n) \text{ (because } n > 100) \\ &= 2n^2 g(n) \end{aligned}$$

   for all $n > 100$. By the definition of Big-O, it follows directly that $n^2 f(n) + g(n)$ is $O(n^2 g(n))$ (where $c = 2$ and $N = 101$). □

5. If $f(n), g(n) > 0$ for all n and $f(n)$ is $O(g(n))$, then $2^{f(n)}$ is $O(2^{g(n)})$.
If $f(n)$ is $O(g(n))$, then there exist c and N such that for all $n \geq N$, $f(n) \leq cg(n)$. We want to prove that there exist c' and N' such that for all $n \geq N'$, $2^{f(n)} \leq c'2^{g(n)}$. Now, $2^{f(n)} \leq 2^{cg(n)}$. Now if $c > 1$, we obviously cannot continue to the conclusion that $2^{f(n)} \leq c'2^{g(n)}$. For example, let $f(n) = 2n$ and $g(n) = n$. Then $f(n)$ is $O(g(n))$ ($N = 1$, $c = 2$), but $2^{f(n)} = 2^{2n} = 4^n$ is not $O(2^n)$. Hence, we have found a counterexample, and therefore the statement is false. □

6. If $f(n), g(n) > 0$ for all n and $\log f(n)$ is $O(\log g(n))$, then $f(n)$ is $O(g(n))$.
This statement is false, using a similar reasoning as in Exercise 5. Consider a counterexample where $f(n) = n^2$ and $g(n) = n$. Then $\log f(n) = \log n^2 = 2 \log n$ is obviously $O(\log n)$.□